MTH 203: Differential Geometry of Curves and Surfaces
Semester 2, 2018-19

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Contents

1 Curves

This section is based on Chapters 1-3 from [2].

1.1 Parametrized curves in $\mathbb{R}^n$

(i) A curve in $C \subset \mathbb{R}^n$ is defined by a set

$$\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : f_i(x) = c_i, \text{ for } 1 \leq i \leq n-1\},$$

where each $f_i : \mathbb{R}^n \to \mathbb{R}$ is a continuous functions, and $c_i \in \mathbb{R}$.

(ii) Examples of curves in $\mathbb{R}^2$.

(a) The parabola $C_1 = \{(x, y) \in \mathbb{R}^3 : x^2 - y = 0\}$.

(b) The circle $C_2 = \{(x, y) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$.

(c) The astroid $C_3 = \{(x, y) \in \mathbb{R}^3 : x^{2/3} + y^{2/3} = 1\}$.

(iii) A *parametrized curve* in $\mathbb{R}^n$ is a continuous map

$$\gamma : (\alpha, \beta) \to \mathbb{R}^n : t \stackrel{\gamma}{\longmapsto} (\gamma_1(t), \ldots, \gamma_n(t))$$

where $-\infty \leq \alpha < \beta \leq \infty$, and the $\gamma_i : \mathbb{R} \to \mathbb{R}$ are continuous maps.

(iv) Examples of parametrized curves.

(a) A parametrization for the curve C_1 is

$$\gamma_1 : (-\infty, \infty) \to \mathbb{R}^2 : t \stackrel{\gamma_1}{\longmapsto} (t, t^2).$$

(b) A parametrization for the curve C_2 is

$$\gamma_2 : (-\infty, \infty) \to \mathbb{R}^2 : t \stackrel{\gamma_2}{\longmapsto} (\cos(t), \sin(t)).$$

(c) A parametrization for the curve C_3 is

$$\gamma_3 : (-\infty, \infty) \to \mathbb{R}^2 : t \stackrel{\gamma_3}{\longmapsto} (\cos^3(t), \sin^3(t)).$$

(v) A parametrized curve $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ is said to be *smooth* is all the derivative $\frac{d^k \gamma_i}{dt^k}$, for $1 \leq i \leq n$ and $k \in \mathbb{N}$ exist and are continuous. From here on, we will assume that all parametrizations are smooth.

(vi) Given a parametrized curve $\gamma : (\alpha, \beta) \to \mathbb{R}^n$, we define its *tangent vector* by

$$\gamma'(t) = \frac{d\gamma}{dt}.$$

(vii) If the tangent vector of a parametrized curve is constant, then the curve is a part of a straight line.

(viii) The *arc length* of a parametrized curve $\gamma(t)$ starting at $\gamma(t_0)$ is defined by the function

$$s(t) = \int_{t_0}^{t} \|\gamma'(u)\| du.$$

(ix) Let $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ be a parametrized curve. Then the *speed* of γ at $\gamma(t)$ is defined by $\|\gamma'(t)\|$. The curve γ is said to be of *unit speed* if $\|\gamma'(t)\| = 1$, for all $t \in (\alpha, \beta)$.

(x) Let $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ be a parametrized curve of unit speed. Then, either $\gamma'' = 0$, or $\gamma'' \perp \gamma'$.

(xi) A parematrized curve $\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \to \mathbb{R}^n$ is said to be a *reparametrization* of a parametrized curve $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ if there exists a smooth bijective map $\phi : (\tilde{\alpha}, \tilde{\beta}) \to (\alpha, \beta)$ such that ϕ^{-1} is smooth, and $\tilde{\alpha}(\tilde{t}) = \gamma(\phi(t))$, for all $\tilde{t} \in (\tilde{\alpha}, \tilde{\beta})$.

(xii) If $\tilde{\gamma}$ is a reparametrization of γ, then γ is a reparametrization of $\tilde{\gamma}$ via the map ϕ^{-1}.

   (a) For example, the curve $\tilde{\gamma}(t) = (\sin(t), \cos(t))$ is a reparametrization of $\gamma(t) = (\cos(t), \sin(t))$ via $\phi(t) = \pi/2 - t$.

1.2 Regular curves

(i) A point $\gamma(t)$ of a parametrized curve γ is said to *regular* if $\gamma'(t) = 0$, and is said to be a *singular* point, otherwise.

(ii) A parametrized curve $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ is said to *regular*, if $\gamma(t)$ is a regular point, for every $t \in (\alpha, \beta)$.

(iii) Examples of regular (or non-regular) curves.

(a) The logarithmic spiral $\gamma(t) = (e^t \cos(t), e^t \sin(t))$ is regular, as $\|\gamma'(t)\|^2 = 2e^{2t} \neq 0$.

(b) The twisted cubic $\gamma(t) = (t, t^2, t^3)$, $t \in (-\infty, \infty)$ is regular, as $\|\gamma'(t)\| = \sqrt{1 + 4t^2 + 9t^4} \neq 0$.

(c) The regularity of a curve is dependent in its parametrization. For example, $\gamma(t) = (t^3, t^6)$ is a not a regular parametrization of the curve $y = x^2$.

(iv) Any reparametrization of a regular curve is regular.

(v) If $\gamma(t)$ is a regular curve, then its arc length $s(t)$ starting at any point of γ is a smooth function of t.

(vi) A reparametrized curve is of unit speed if, and only if, its regular.

(vii) Let γ be a regular curve, and let $\tilde{\gamma}$ be a reparametrization of γ given by $\tilde{\gamma}(u(t)) = \gamma(t)$, where u is a smooth function of t. Then $\tilde{\gamma}$ is of unit speed if, and only if,

$$u(t) = \pm s(t) + c,$$

where $s(t)$ is a the arc length and c is a constant.

(viii) Example of reparametrizations.

(a) The curve $\gamma(t) = e^t \cos(t), e^t \sin(t)$ has arc length $s(t) = \sqrt{2}(e^t - 1)$, and a unit-speed reparametrization given by $t = \log(s/\sqrt{2} + 1)$.

(b) The curve $\gamma(t) = (t, t^2, t^3)$ has arc-length given by the elliptic integral

$$s(t) = \int_0^t \sqrt{1 + 4u^2 = 9u^4}\, du.$$

(ix) The *level set* of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ is a set of the form

$$\{x \in \mathbb{R}^n : f(x) = c\},$$

where $c \in \mathbb{R}$. A level set of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ is called a *level curve.*

(x) Let $f(x, y)$ be a smooth function in two variables. Assume that, at every point of the level curve $C = \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}$, the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial x}$ are not both zero. If $P(x_0, y_0)$ is a point of C, there exists a regular parametrized curve $\gamma(t)$ defined on an open interval containing 0 such that $\gamma(0) = (x_0, y_0)$, and $\gamma(t) \in C$, for all t.

(xi) Let γ be a regular parametrized curve in $\mathbb{R}^2$, and let $\gamma(t_0) = (x_0, y_0)$. Then there exists a smooth real-valued function $f(x, y)$ defined for all points x and y defined in open intervals containing x_0 and y_0, respectively, (satisfying the conditions of (x) above) such that $\gamma(t) \in \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}$, for all t in some open interval containing t_0.

1.3 Curvature of curves

(i) Let γ be a unit speed curve with parameter s, and let $\dot{\gamma} = \frac{d\gamma}{ds}$. Then the *curvature* of γ at a point $\gamma(s)$ is defined by

$$\kappa(s) = \|\ddot{\gamma}(s)\|.$$

(ii) Examples of curvature.

(a) The curvature of a line is zero.

(b) The curvature of a circle $\gamma(s) = x_0 + R\cos(s/R)) + y_0 + R\sin(s/R))$ in $\mathbb{R}^2$ with center (x_0, y_0) and radius R is given by $\kappa = 1/R$.

(iii) The curvature of a curve remains invariant under reparametrization.

(iv) Let γ be a regular curve in $\mathbb{R}^3$ with parameter t. Then its curvature is given at the point $\gamma(t)$ is given by

$$\kappa(t) = \frac{\|\gamma''(t) \times \gamma'(t)\|}{\|\gamma'(t)\|^3},$$

where $\gamma'(t) = \frac{d\gamma}{dt}$.

(v) For example, the curvature of the helix h about z-axis

$$h(\theta) = (a\cos(\theta), a\sin(\theta), b\theta), \ -\infty < \theta < \infty$$

is given by $\kappa = |a|/(a^2 + b^2)$.

1.4 Plane curves

(i) Let γ be a unit-speed plane curve with parameter s, and let $T(s)$ denote the unit tangent vector at $\gamma(s)$.

(a) The *signed unit normal* $n(s)$ to $\gamma(s)$ (at $\gamma(s)$) is the unit vector obtained by rotating $T(s) = \dot{\gamma}(s)$ counter-clockwise by $\pi/2$.

(b) Since $\ddot{\gamma}(s)$ is parallel to $n(s)$, it follows that $\ddot{\gamma}(s) = \kappa_{\pm}(s)n(s)$, where $\kappa_{\pm}(s)$ is called the *signed curvature* of γ. By definition, we have

$$\kappa(s) = \|\ddot{\gamma}(s)\| = \|\kappa_{\pm}(s)n(s)\| = |\kappa_{\pm}(s)|.$$

(ii) Let γ be a unit-speed plane curve with parameter s, and let $\varphi(s)$ be the angle through which a fixed unit vector must be rotated counter-clockwise to bring it into coincidence with the unit tangent vector T. Then

$$\kappa_{\pm}(s) = \frac{d\varphi}{ds}.$$

In particular, the signed curvature of a curve is the rate of rotation of its tangent vector.

(iii) Let $\kappa : (\alpha, \beta) \to \mathbb{R}$ be a smooth function. Then there exists a unit-speed curve $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ whose signed curvature is κ given by

$$\gamma(s) = \left(\int_{s_0}^{s} \cos(\varphi(t))dt, \int_{s_0}^{s} \sin(\varphi(t))dt \right), \text{ where } \varphi(s) = \int_{s_0}^{s} \kappa(u)du.$$

Furthermore, if $\tilde{\gamma} : (\alpha, \beta) \to \mathbb{R}^2$ is another unit-speed curve whose signed curvature is κ, then there exists a rigid motion M of $\mathbb{R}^2$ such that

$$\tilde{\gamma}(s) = M(\gamma(s)), \text{ for all } s \in (\alpha, \beta).$$

(iv) Examples of signed curvature.

(a) The signed curvature of a circle $\gamma(s) = x_0 + R\cos(s/R)) + y_0 + R\sin(s/R))$ in $\mathbb{R}^2$ is given by $\kappa_{\pm} = 1/R$.

(b) By (iii), a plane curve whose signed curvature is $\kappa_{\pm}(s) = s$ is given by the Fresnel's integrals

$$\gamma(s) = \left(\int_{0}^{s} \cos(t^2/2)dt, \int_{0}^{s} \sin(t^2/2)dt \right).$$

(v) Any regular plane curve whose curvature is a positive constant is a part of a circle.

1.5 Space curves

(i) Let γ be a unit-speed curve in $\mathbb{R}^3$ with parameter s.

(a) Assuming that $k(s) \neq 0$, for any s, we define the *principal normal* of γ at $\gamma(s)$ to be

$$\eta(s) = \frac{1}{\kappa(s)} \frac{dT}{ds}.$$

(b) We define the *binormal vector* of γ at $\gamma(s)$ to be

$$b(s) = T(s) \times \eta(s).$$

(ii) The unit-vectors $T(s), \eta(s)$, and $b(s)$, form an orthonormal basis for $\mathbb{R}^3$, for every s.

(iii) At every point $\gamma(s)$ in a unit-speed space curve γ, $\dot{b}(s) = -\tau(s)\eta(s)$, where $\tau(s)$ is a scalar called the *torsion* of γ. As τ remains invariant under reparametrization, we define the torsion of an arbitrary regular curve γ to be the torsion of the unit-speed reparametrization of γ.

(iv) Let $\gamma(t)$ be a regular curve with nowhere-vanishing curvature. Then

$$\tau = \frac{(\gamma' \times \gamma'') \cdot \gamma'''}{\|\dot{\gamma} \times \ddot{\gamma}\|^2},$$

where $\gamma' = \frac{d\gamma}{dt}$ and $\dot{\gamma} = \frac{d\gamma}{ds}$.

(v) Let γ be a regular space curve with nowhere-vanishing curvature. Then $\gamma(t)$ is contained in a plan (i.e. planar) if, and only if, $\tau = 0$, at every point in γ.

(vi) (Serret-Frenet) Let γ be a unit-speed space curve with nowhere-vanishing curvature. Then

$$\begin{bmatrix} \dot{T} \\ \eta \\ b \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ \eta \\ b \end{bmatrix}.$$

(vii) Let γ be a unit-speed space curve with constant curvature and zero torsion. Then γ is a part of a circle.

(viii) Let $\kappa, \tau : \mathbb{R}^3 \to \mathbb{R}$ be smooth functions with $\kappa > 0$ everywhere. Then there exists a unit-speed curve γ in $\mathbb{R}^3$ whose curvature is κ and torsion is τ. Moreover, if $\tilde{\gamma}$ is another curve in $\mathbb{R}^3$ with curvature κ and torsion τ, then there exists a rigid motion M (of $\mathbb{R}^3$) such that

$$\tilde{\gamma}(s) = M(\gamma(s)), \text{ for all } s.$$

1.6 Simple closed curves

(i) Let $k \in \mathbb{R}$ be a positive constant. A *simple closed plane curve with period* k is a regular curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ such that

$$\gamma(t) = \gamma(t') \iff t - t' = nk, \text{ where } n \in \mathbb{Z}.$$

(ii) For example, the parametrized circle $\gamma(t) = \cos(2\pi t/k), \sin(2\pi t/k)$ is a simple closed curve of period k.

(iii) (Jordan Curve Theorem) Any simple closed plane curve γ has an interior $\mathrm{int}(\gamma)$ and an exterior $\mathrm{ext}(\gamma)$ such that:

(a) $\mathrm{int}(\gamma)$ is bounded,

(b) $\mathrm{ext}(\gamma)$ is unbounded, and

(c) both $\mathrm{int}(\gamma)$ and $\mathrm{ext}(\gamma)$ are connected.

(iv) Since every point $\gamma(t)$ in a simple closed plane curve γ of period k is traced out by an interval of length k , we may assume (without loss of generality) that

$$\gamma : [0, k] \to \mathbb{R}^2.$$

(v) The *length* $\ell(\gamma)$ of a simple closed plane curve $\gamma : [0, k] \to \mathbb{R}^2$ of period k is given by

$$\ell(\gamma) = \int_0^a \|\gamma'(t)\| dt.$$

(vi) Let $\gamma : [0, k] \to \mathbb{R}^2$ be a simple closed plane curve of period k. Then a unit-speed reparameterization $\tilde{\gamma}$ of γ has period $\ell(\gamma)$.

(vii) A simple closed plane curve is said to be *positively oriented* if its signed unit normal $n(s)$ points inward toward $\text{int}(\gamma)$ at every point in $\gamma(s)$ in γ. As a convention, we shall assume from here on that all simple closed curves are positively oriented.

(viii) If $\gamma(t) = (x(t), y(t))$ be a positively oriented simple closed plane curve with period k. Then the *area of the interior of* γ is given by

$$\text{Area}(\text{int}(\gamma)) = \frac{1}{2}\int_0^k (xy' - yx')dt.$$

(ix) (Wirtinger's Inequality) Let $F : [0, \pi] \to \mathbb{R}$ be a smooth function such that $F(0) = F(\pi) = 0$. Then

$$\int_0^\pi \left(\frac{dF}{dt}\right)^2 dt \geq \int_0^\pi F(t)^2 dt,$$

with equality holding if, and only if, $F(t) = A\sin(t)$, for all $t \in [0, \pi]$, where A is a constant.

(x) (Isoperimetric inequality) Let γ be a simple closed plane curve. Then

$$\text{Area}(\text{int}(\gamma)) \leq \frac{1}{4\pi}\ell(\gamma)^2,$$

with equality holding if, and only if, γ is a circle.

(xi) The *vertex* of a plane curve γ is a point where its signed curvature $\kappa_{\pm}$ has a stationary point (i.e $\frac{d\kappa_{\pm}}{dt} = 0$).

(xii) For example, the curve $\gamma(t) = (a\cos(t), b\sin(t))$ has vertices at $t = 0,\ \pi/2,\ \pi,$ and $3\pi/2$.

(xiii) A simple closed plane curve γ is said to be *convex* if for any two points $P, Q \in \text{int}(\gamma)$, the straight line segment joining P and Q lies inside $\text{int}(\gamma)$.

(xiv) (Four-vertex theorem) Every convex simple closed plane curve has at least four vertices.

2 Surfaces

This section is based on Chapters 2-3 from [1] and Chapters 3-4 from [3].

2.1 Regular surfaces

(i) A subset $S \subset \mathbb{R}^3$ is called *regular surface* if, for each $p \in S$, there exists a neighborhood $V \ni p$, and a map $f : U \to V \cap S$ of an open set $U \in \mathbb{R}^2$ onto $V \cap S$ such that:

(1) f is differentiable, that is,

$$f(u, v) = (x(u, v), y(u, v), z(u, v))$$

has continuous partials of all orders,

(2) f is a homeomorphism, and

(3) for each $q \in U$, the differential $df_q : \mathbb{R}^3 \to \mathbb{R}^3$ is injective, that is, at least one of the Jacobians

$$\frac{\partial(x, y)}{\partial(u, v)}, \frac{\partial(y, z)}{\partial(u, v)}, \frac{\partial(z, x)}{\partial(u, v)}$$

is nonzero at $q \in U$.

The map f is called a *parametrization or local coordinates* at p, and the neighborhood $V \cap S \ni p$ is called a *coordinate neighborhood at p.*

(ii) Examples of regular surfaces.

(a) The unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$ is a regular surface with the parametrizations

$$f_1^{\pm} : \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \to S^2 : (x, y) \xmapsto{f_1^{\pm}} (x, y, \pm\sqrt{1 - x^2 - y^2})$$
$$f_2^{\pm} : \{(y, z) \in \mathbb{R}^2 : y^2 + z^2 < 1\} \to S^2 : (y, z) \xmapsto{f_2^{\pm}} (\pm\sqrt{1 - y^2 - z^2}, y, z)$$
$$f_3^{\pm} : \{(x, z) \in \mathbb{R}^2 : x^2 + z^2 < 1\} \to S^2 : (x, z) \xmapsto{f_3^{\pm}} (x, \pm\sqrt{1 - x^2 - z^2}, z)$$

The parametrizations above can also be described in the ususal spherical coordinates given by

$$f(\theta, \varphi) = (\sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi), \cos(\theta)),\ \text{where } 0 < \theta < \pi \text{ and } 0 < \varphi < 2\pi.$$

(b) The torus T^2 is the surface generated by rotating a circle of radius r about a straight line belonging to the plane of the circle and at a distance $a > r$ away from the center of the circle. T^2 is a regular surface with a parametrization given by

$$f(u,v) = ((r\cos(u)+a)\cos(v), (r\cos(u)+a)\sin(v), r\sin(v)),$$
$$\text{where } 0 < u < 2\pi \text{ and } 0 < v < 2\pi.$$

(iii) Let $U \subset \mathbb{R}^2$ be an open set, and let $f : U \to \mathbb{R}$ be a differentiable map. Then the graph of f is a regular surface.

(iv) Let $U \subset \mathbb{R}^n$ be an open set, and let $f : U \to \mathbb{R}^m$ be a differentiable map.

(a) A point $p \in U$ is called a *critical point* if $df_p : \mathbb{R}^n \to \mathbb{R}^m$ is not surjective.

(b) The image $f(p)$ of a critical point p is called a *critical value.*

(c) A point in $f(U)$ that is a not a critical value is called a *regular value.*

(v) Let $U \subset \mathbb{R}^3$ be an open set, and let $f : U \to \mathbb{R}$ be a differentiable map. If $q \in f(U)$ is a regular value, then $f^{-1}(q)$ is a regular surface.

(vi) Let $S \subset \mathbb{R}^3$ be a regular surface, and let $p \in S$. Then there exists a neighborhood $V \ni p$ in S such that V is the graph of a differentiable function which has one of the following three forms:

$$z = f(x,y),\ y = g(x,z),\ \text{and } x = h(y,z).$$

(vii) Examples and nonexamples of regular surfaces.

(a) The torus T^2 in (ii)(b) is given by the equation

$$z^2 = r^2 - (\sqrt{x^2+y^2} - a)^2.$$

Note that the function

$$f(x,y,z) = z^2 + (\sqrt{x^2+y^2} - a)^2$$

is differentiable, whenever $(x,y) \neq (0,0)$. Thus, r^2 is a regular value of f, and by (v), it follows that $T^2 = f^{-1}(r^2)$ is a regular surface.

(b) The one-sheeted cone C given by the equation $z = \sqrt{x^2 + y^2}$ is not a regular surface for the following reason. If C were regular, then by (vi), C must be the graph of a diffrentiable function of the form $z = f(x, y)$, which has to agree with $z = \sqrt{x^2 + y^2}$ in a neighborhood of $(0, 0, 0)$. But this is impossible, as $z = \sqrt{x^2 + y^2}$ is not differentiable at $(0, 0)$.

(viii) Let S be a regular surface, and let $p \in S$. If $f : U \to \mathbb{R}^3$, where $U \subset \mathbb{R}^3$ is open, be an injective map with $p \in f(U) \subset S$ such that conditions (1) and (3) of (i) hold, then f^{-1} is continuous.

2.2 Change of coordinates

(i) Let S_1, S_2 be regular surfaces, and let $V_1 \subset S_1$ be an open set. A continuous map $\varphi : V \to S_2$ is said to be *differentiable* at $p \in V_1$ if, given parametrizations

$$f_i : U_i \to S_i, \text{ for } i = 1, 2$$

with $p \in f_1(U)$ and $\varphi(f_1(U_1)) \subset f_2(U_2)$, the map $f_2^{-1} \circ \varphi \circ f_1 : U_1 \to U_2$ is differentiable at $q = f_1^{-1}(p)$.

(ii) Two regular surfaces S_1 and S_2 are said to be *diffeomorphic* if there exists a differentiable map $\varphi : S_1 \to S_2$ with a differentiable inverse $\varphi^{-1} : S_2 \to S_1$. Such a map φ is called a *diffeomorphism* from S_1 to S_2.

(iii) Example of differentiable maps and diffeomorphisms.

(a) If $f : U(\subset \mathbb{R}^2) \to S$ is a parametrization, then $f : U \to f(U)$ is a diffeomorphism.

(b) Let S_i, for $i = 1, 2$ be regular surfaces, let $S_1 \subset V$, where, where V is an open set of $\mathbb{R}^3$. If $\varphi : V \to \mathbb{R}^3$ is a differentiable map such that $\varphi(S_1) \subset S_2$, then $\varphi|_{S_1} : S_1 \to S_2$ is differentiable. In particular, the following maps are differentiable.

(1) Let $S \subset \mathbb{R}^3$ be a surface that is symmetric about the xy-plane. Then its reflection about the xy-plane given by

$$\rho : S \to S : (x, y, z) \overset{\rho}{\mapsto} (x, y, -z)$$

is a diffeomorphism.

(2) Let $R_{z,\theta}$ be a rotation in $\mathbb{R}^3$ about the z-axis counter-clockwise by θ radians. If S is a regular surface such that $R_{z,\theta}(S) \subset S$, then $R_{z,\theta}|_S$ is a diffeomorphism.

(3) For fixed nonzero real numbers a, b, c, the differentiable map

$$\varphi : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \overset{\varphi}{\mapsto} (xa, yb, zc)$$

restricts to diffeomorphism from the sphere S^2 onto the ellipsoid $\{(x, y, z) \in \mathbb{R}^3 : (x/a)^2 + (y/b)^2 + (z/c)^2 = 1\}$.

(iv) Let S be a regular surface, and let $p \in S$. Let $f : U(\subset \mathbb{R}^2) \to S$ and $g : V(\subset \mathbb{R}^2) \to S$ be two parametrizations of S such that $p \in x(U) \subset y(V) = W$. Then the *change of coordinates* defined by

$$h := f^{-1} \circ g : g^{-1}(W) \to g^{-1}(W)$$

is a diffeomorphism.

(v) Let S be a regular surface, let $V \subset \mathbb{R}^3$ be a open set such that $S \subset V$. If $f : V \to \mathbb{R}$ is a differentiable function, then so is $f|_S$.

(vi) Examples of differentiable functions on a regular surface S.

(a) For a fixed unit vector $v \in \mathbb{R}^3$, the *height function relative of* v defined by

$$h_v : S \to \mathbb{R} : w \overset{h_v}{\longmapsto} w \cdot v$$

is differentiable function on S.

(b) For a fixed unit vector $p_0 \in \mathbb{R}^3$, the function defined by

$$f : S \to \mathbb{R} : p \overset{f}{\mapsto} \|p - p_0\|^2$$

is differentiable function on S.

(vii) Let $f : U(\subset \mathbb{R}^2) \to \mathbb{R}^3$ be a differentiable map. Then:

(a) The map f is called a *parametrized surface*.

(b) The set $f(U)$ is called the *trace* of f.

(c) The map f is said to be *regular* if the differential $df_q : \mathbb{R}2 \to \mathbb{R}^3$ is injective for all $q \in U$.

(d) A point $p \in U$ where df_p is not injective is called a *singular point* of f.

(viii) Two important examples of parametrized surfaces.

(a) (Surface of revolution) Let $S \subset \mathbb{R}^3$ be the surface obtained by rotating a regular connected plane curve C about an axis ℓ in the plane which does not intersect the curve. Then S is called the *surface of revolution* generated by the curve C with *rotation axis* ℓ. In particular, let C be a curve in the xz-plane with a parametrization

$$x = f(v),\ z = g(v),\ a < v < b \text{ and } f(v) > 0,$$

that is rotated about the z-axis. Then S is a regular parametrized surface with a parametrization given by $F : U \to \mathbb{R}^3$, where

$$F(u, v) = (f(v)\cos(u), f(v)\sin(u), g(v)),$$

and

$$U = \{(u, v) \in \mathbb{R}^2 : 0 < u < 2\pi \text{ and } a < v < b\}.$$

(b) Let $\alpha : I \to \mathbb{R}^3$ be a non-planar parametrized curve. Then

$$f_\alpha(x, y) = \alpha(x) + y\alpha'(x),\ (x, y) \in I \times \mathbb{R}$$

is a parametrized surface called the *tangent surface*. Restricting the domain of f to $U = \{(t, v) \in I \times R : v \neq 0\}$, we see that $f : U \to \mathbb{R}^3$ is a regular surface whose trace has two connected components with a common boundary $\alpha(I)$.

(c) Let $f : U \to \mathbb{R}^3$ be a regular parametrized surface, and let $q \in U$. Then there exists a neighborhood $V \ni q$ in $\mathbb{R}^2$ such that $f(V) \subset \mathbb{R}^3$ is a regular surface.

2.3 Tangent space

(i) Let S be a regular surface. A *tangent vector* to S at a point $p \in S$ is the tangent vector $\alpha'(0)$ of differentiable paramterized curve $\alpha : (\epsilon, \epsilon) \to S$ with $\alpha(0) = p$.

(ii) Let $f : U(\subset \mathbb{R}^n) \to \mathbb{R}^m$ be a differentiable map. To each $p \in U$, we associate a linear map $df_p : \mathbb{R}^n \to \mathbb{R}^m$ called the *differential of f at p* which is defined as follows. Let $w \in \mathbb{R}^n$, and let $\alpha : (-\epsilon, \epsilon) \to U$ be a differentiable curve such that $\alpha(0) = p$ and $\alpha'(0) = w$. Then $\beta = f \circ \alpha$ is differentiable, and we define

$$df_p(w) := \beta'(0).$$

(iii) Let $f : U(\subset \mathbb{R}^2) \to S$ be a parametrization of a regular surface S, and let $q \in U$. Then the vector space $df_q(\mathbb{R}^2) \subset \mathbb{R}^3$ coincides with the set of tangent vectors to S at $f(q)$.

(iv) Let $f : U(\subset \mathbb{R}^2) \to S$ be a parametrization of a regular surface S, and let $q \in U$. Then the plane $df_q(\mathbb{R}^2)$ is called the *tangent plane to S at $p = f(q)$* denoted by $T_p(S)$. Moreover, the parametrization f determines a choice of basis $\{f_u(q), f_v(q)\}$ of $T_p(S)$ called the *basis associated with f*.

(v) Let $f : U(\subset \mathbb{R}^2) \to S$ be a parametrization of a regular surface S, and let $q \in U$. Let $w = \alpha'(0)$, where $\alpha = f \circ \beta$ and $\beta : (-\epsilon, \epsilon) \to U$ is $\beta(t) = (u(t), v(t))$, $\beta(0) = q = f^{-1}(p)$. Then

$$\alpha'(0) = f_u(q)u'(0) + f_v(q)v'(0) = w.$$

(vi) Let S_1, S_2 be regular surfaces, and let $\varphi = (\varphi_1, \varphi_2) : V(\subset S_1) \to S_2$ be a differentiable map. Consider $w \in T_p(S_1)$ such that $w = \alpha'(0)$, where $\alpha : (-\epsilon, \epsilon) \to V : t \overset{\alpha}{\mapsto} (u(t), v(t))$ with $\alpha(0) = p$. Further, assume that the curve $\beta = \varphi \circ \alpha$ is such that $\beta(0) = \varphi(p)$ ($\implies \beta'(0) \in T_{\varphi(p)}(S_2)$). Then the map $d\varphi_p : T_p(S_1) \to T_{\varphi(p)}(S_2)$ defined by $d\varphi_p(w) = \beta'(0)$ is linear define by

$$\beta'(0) = d\varphi_p(w) = \begin{bmatrix} \frac{\partial \varphi_1}{\partial u} & \frac{\partial \varphi_1}{\partial v} \\ \frac{\partial \varphi_2}{\partial u} & \frac{\partial \varphi_2}{\partial v} \end{bmatrix} \begin{bmatrix} u'(0) \\ v'(0) \end{bmatrix}.$$

(vii) Examples of differentials of maps.

(a) For a fixed unit vector $v \in \mathbb{R}^3$, the differential of the height function relative of v defined by $h_v : S \to \mathbb{R} : w \overset{h_v}{\longmapsto} w \cdot v$ at $p \in S$ is given by

$$(dh_v)_p(w) = w \cdot v, \forall\, w \in T_p(S).$$

(b) The differential of the rotation $R_{z,\theta}$ in $\mathbb{R}^3$ about the z-axis by θ radians restricted to S^2 has a differential at $p \in S^2$ given by

$$(dR_{z,\theta})_p(w) = R_{z,\theta}(w), \, \forall\, w \in T_p(S^2).$$

(viii) Let S_1 and S_2 be regular surfaces, and let $\varphi : U(\subset S_1) \to S_2$ be a differentiable map of an open set $U \subset S_1$ such that $d\varphi_p$ is an isomorphism at $p \in U$. Then φ is a local diffeomorphism at p.

2.4 Orientation

(i) Let V be a vector space of dimension 2 over $\mathbb{R}$.

(a) An *orientation* for V is a choice of unit-length normal vector N to V.

(b) With respect a fixed orientation N for V, an ordered basis $\{v_1, v_2\}$ basis for V is said to be *positively oriented* if

$$\frac{v_1 \times v_2}{\|v_1 \times v_2\|} = N.$$

(c) With respect a fixed orientation N for V, an ordered basis $\{v_1, v_2\}$ basis for V is said to be *negatively oriented* if

$$\frac{v_1 \times v_2}{\|v_1 \times v_2\|} = -N.$$

(ii) Let V, W be a vector spaces of dimension 2 over $\mathbb{R}$. Then an isomorphism $T : V \to W$ is said to be *orientation-preserving* if for any positively oriented ordered basis $\{v_1, v_2\}$ for V, $\{T(v_1), T(v_2)\}$ is a positively oriented ordered basis for W.

(iii) Let V, W be a vector spaces of dimension 2 over $\mathbb{R}$. Then an isomorphism $T : V \to W$ is orientation-preserving if, and only if, the matrix of T with respect to any (choice of) positively oriented ordered bases for V and W has positive determinant.

(iv) Let S be a regular surface. A *unit normal* vector N_p to S at $p \in S$ is a unit-length normal vector to $T_p(s)$, that is, $\langle N_p, v\rangle = 0$, for all $v \in T_p(S)$.

(v) Let S be a regular surface.

   (a) A *vector field* on S is a smooth map $F : S \to \mathbb{R}^3$.

   (b) A vector $F : S \to \mathbb{R}^3$ on S is said to be a *unit normal field* if $T(p) = N_p$, for all $p \in S$, where N_p is the unit vector to S at p.

(vi) Let S be a regular surface.

   (a) An *orientation* for S is a unit normal field on S.

   (b) If S has a orientation, then S is said to be *orientable*.

   (c) If S has no orientation, then S is said to be *nonorientable*.

   (d) Let S be orientable. Then S together with a choice of orientation on it is called an *oriented surface*.

(vii) Examples of orientable (and nonorientable) surfaces.

   (a) For a fixed nonzero unit vector $v \in \mathbb{R}^3$, the plane $P_v \subset \mathbb{R}^3$ with unit normal v is given by

$$P_v = \{w \in \mathbb{R}^3 : w \cdot v = const\}.$$

   Then P_v is a regular orientable surface with an orientation F on P_v defined by $F(w) = v$, for all $w \in P_v$.

   (b) Let S be a regular surface that is realized as the level surface of a smooth map $f : \mathbb{R}^3 \to \mathbb{R}$, that is, $S = f^{-1}(\lambda)$, for some regular value λ of f. The S is an orientable surface with a natural orientation $F : S \to \mathbb{R}^3$ given by

$$F(p) = \frac{\nabla f(p)}{\|\nabla f(p)\|}, \forall\, p \in S.$$

   (c) It follows from (b) that the sphere S^2 is a orientable surface with an orientation given by $F(p) = p$, for all $p \in S^2$.

   (d) Let $G_f = \{(x, y, f(x, y)) : (x, y) \in U\}$ be the graph of a smooth function $f : U(\subset \mathbb{R}^2) \to \mathbb{R}^3$, where U is open. Then G_f is a orientable surface with an orientation $F : G_f \to \mathbb{R}^3$ given by

$$F(p) = \frac{(-f_x(p), -f_y(p), 1)}{\sqrt{f_x^2(p) + f_y^2(p) + 1}}.$$

(e) The Möbius band M given by the parametrization $f : (0, 2\pi) \times (-1/2, 1/2) \to \mathbb{R}^3$ defined by

$$f(u, v) = (\cos(u)(2+v\sin(u/2)), \sin(u)(2+v\sin(u/2)), v\cos(u/2))$$

is a nonorientable regular surface. To see this, we consider the coordinate neighborhoods $f_i : U_i \to V_i \subset M$, for $i = 1, 2$, where

$$U_1 = \{(u, v) \in \mathbb{R}^2 : (u, v) \in (\pi/3, 5\pi/3) \times (-1/2, 1/2)\}$$
and
$$U_2 = \{(u, v) \in \mathbb{R}^2 : (u, v) \in (4\pi/3, 8\pi/3) \times (-1/2, 1/2)\}.$$

Let N_i be the unit normal fields induced on V_i. It can be shown that there exists no global normal fields N on M such that $N|_{V_i} = N_i$.

(viii) A connected orientable regular surface has exactly two orientations; Each is the negative of the other.

(ix) The *connected sum* $\Sigma_1 \# \Sigma_2$ of 2 regular surfaces Σ_i is formed by deleting open disks $B_i \subset \Sigma_i$ and the identifying the resulting manifolds $\Sigma_i - B_i$ to each other along the resultant boundaries by a homeomorphism $h : \partial\Sigma_2 \to \partial\Sigma_1$ so that

$$\Sigma_1 \# \Sigma_2 = (\Sigma_1 - B_1) \sqcup_h (\Sigma_2 - B_2).$$

(x) For $g \geq 1$, the connected sum of g copies of the torus T^2 is called the *closed orientable surface S_g of genus g.* For $k \geq 1$, the connected sum of k copies of the real projective plane $\mathbb{R}P^2$ is called the *closed nonorientable surface N_k with k crosscaps.*

(xi) (Classification theorem for connected closed regular surfaces) Let S be a be a closed (i.e. $\partial S = \emptyset$) connected regular surface.

(a) If S is orientable, then S is diffeomorphic to the 2-sphere S^2 or S_g, for some $g \geq 1$.

(b) If S is nonorientable, then S is diffeomorphic to N_k, for some $k \geq 1$.

(xii) Let S, S' be connected oriented regular surfaces, and $f : S \to S'$ be a diffeomorphism. Then:

(a) f is said to be *orientation-preserving* if for each $p \in S$, $df_p : T_p(S) \to T_{f(p)}(S')$ is orientation-preserving.

(b) f is said to be *orientation-reversing*, if f is not orientation-preserving.

(xiii) Examples of orientation-preserving (and reversing) diffeomorphisms.

(a) Let $S \subset \mathbb{R}^3$ be a connected regular oriented surface that is left invariant by a rotation $A \in O(3)$ about the origin. Then the diffeomorphism $A|_S : S \to S$ is orientation-preserving if, and only if $\mathrm{Det}(A) = 1$.

(b) Let $S \subset \mathbb{R}^3$ be a connected regular oriented surface that is left invariant by a reflection R_P about a plane in $P \subset \mathbb{R}^3$. Then $R_P|_S : S \to S$ is a orientation-reversing diffeomorphism.

(xiv) Let S, S' be connected oriented regular surfaces, and $f : S \to S'$ be a diffeomorphism. Then f is orientation-preserving if, and only if, there exists a point $p \in S$ such that $df_p : T_p(S) \to T_{f(p)}(S)$ is orientation-preserving.

2.5 Surface area

(i) Let $f : U(\subset \mathbb{R}^2) \to S$ be a surface patch with $q \in U$, and let $p = f(q)$. Choosing the standard basis $\{e_1, e_2\}$ of $\mathbb{R}^2$, the *area distortion* of the linear transformation $df_q : \mathbb{R}^2 \to T_p(S)$ is given by

$$\|f_u(q) \times f_v(q)\|.$$

(ii) Let S be a regular surface, and let $R \subset S$.

(i) We call R a *polygonal region* if R is covered by a single coordinate chart $f : U \to S$ such that $f^{-1}(R)$ equal the union of the interior of a piecewise-regular simple closed curve with its boundary.

(ii) We define the *area (or surface area)* of a polygonal region R by

$$Area(R) = \iint_{f^{-1}(R)} \|f_u \times f_v\| \, dA.$$

(iii) If R is union of finitely many polygonal regions intersecting only along boundaries, then we define the *area of* R as the sum of the areas over all of these polygonal regions.

(iv) We define the *integral over a polygonal region* R of a smooth function $g : R \to \mathbb{R}$ by

$$\iint_R g\,dA = \iint_{f^{-1}(R)} (g \circ f) \cdot \|f_u \times f_v\|\,dA.$$

(v) If $\psi : U_1 \to U_2$ is a diffeomorphism between open sets in $\mathbb{R}^2$. Let $\varphi : U_2 \to \mathbb{R}$ is smooth, and $K \subset U_2$ is a polygonal region. Then

$$\iint_K \varphi\,dA = \iint_{\psi^{-1}(K)} (\varphi \circ \psi) \cdot \|\psi_u \times \psi_v\|\,dA.$$

(iii) Examples of surface areas.

(a) The graph G_f of a smooth function $f : U(\subset \mathbb{R}^2) \to \mathbb{R}$ of covered by a single surface patch $g : U \to G : (x, y) \overset{g}{\mapsto} (x, y, f(x, y))$. Therefore, the area of a polygonal region $R \subset G$ is given by

$$Area(R) = \iint_{g^{-1}(R)} \sqrt{f_x^2 + f_y^2 + 1}\,dA.$$

(b) The graph G_f of a smooth function $f : U(\subset \mathbb{R}^2) \to \mathbb{R}$ of covered by a single surface patch $g : U \to G : (x, y) \overset{g}{\mapsto} (x, y, f(x, y))$. Therefore, the area of a polygonal region $R \subset G$ is given by

$$Area(R) = \iint_{g^{-1}(R)} \sqrt{f_x^2 + f_y^2 + 1}\,dA.$$

(c) The unit sphere S^2 is covered by the coordinate chart

$$f : (0, 2\pi)\times(0, \pi) \to S^2 : (\theta, \phi) \overset{f}{\mapsto} (\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)).$$

Consequently,

$$Area(S^2) = 4\pi.$$

(d) If $f : S_1 \to S_2$ is a diffeomorphism between regular surfaces and $R \subset S_1$ is a polygonal region, then $f(R)$ is a polygonal region, and

$$Area(f(R)) = \iint_R |f_u \times f_v|\,dA.$$

2.6 Isometries and the first fundamental form

(i) The *first fundamental form* of a regular surface S assigns to each $p \in S$ the quadratic form

$$I_p : T_p(S) \to \mathbb{R} : x \overset{I_p}{\longmapsto} \|x\|_p^2,$$

where $\|x\|$ is the usual square norm in $\mathbb{R}^2$.

(ii) Let $f : U(\subset \mathbb{R}^2) \to S$ be a surface patch. A tangent vector $w \in T_p(S)$ is the tangent vector to a parametrized curve $\alpha(t) = f(u(t), v(t))$, where $t \in (-\epsilon, \epsilon)$ with $p = \alpha(0) = f(u_0, v_0)$. Consequently, we have

$$I_p(\alpha'(0)) = E(u')^2 + Fu'v' + G(v')^2,$$

where

$$E(u_0, v_0) = \langle f_u, f_u \rangle_p, \ F(u_0, v_0) = \langle f_u, f_v \rangle_p, \text{ and } G(u_0, v_0) = \langle f_v, f_v \rangle_p.$$

(iii) Let S_1, S_2 be regular surfaces . A diffeomorphism $f : S_1 \to S_2$ is called an *isometry* if df preserves their first fundamental forms, that is,

$$\|df_p(x)\|_{f(p)}^2 = \|x\|_p^2, \ \forall p \in S_1 \text{ and } x \in T_p(S_1)$$

This is equivalent to saying that df preserves the inner products of S_1 and S_2, that is,

$$\langle df_p(x), df_p(y) \rangle_{f(p)} = \langle x, y \rangle_p, \ \forall p \in S_1 \text{ and } x, y \in T_p(S_1).$$

(iv) Examples of isometries.

(a) If f is a rigid motion of $\mathbb{R}^3$ and S is a regular surface, $f(S)$ is a regular surface and $f|_S : S \to f(S)$ is an isometry.

(b) The cylindrical patch

$$f : (-\pi, \pi) \times \mathbb{R} \to C : (u, v) \overset{f}{\longmapsto} (\cos(u), \sin(u), v)$$

is an isometry.

2.7 Conformal and equiareal maps

(i) Let S_1, S_2 be regular surfaces. A diffeomorphism $f : S_1 \to S_2$ is called *equiareal* if

$$\|f_u(p) \times f_v(p)\| = 1, \, \forall p \in S_1.$$

(ii) A diffeomorphism $f : S_1 \to S_2$ is equiareal if, and only if preserves the areas of polygonal regions.

(iii) Let S_1, S_2 be a regular surfaces. A diffeomorphism $f : S_! \to S_2$ is called *conformal* if df is angle-preserving, that is,

$$\angle(x, y) = \angle(df_p(x), df_p(y)), \, \forall p \in S_1 \text{ and } x, y \in T_p(S_1).$$

(iv) A diffeomorphism $f : S_1 \to S_2$ is conformal if, and only if, there exists a smooth positive-valued function $\lambda : S_1 \to \mathbb{R}$ such that

$$\langle df_p(x), df_p(y)\rangle_{f(p)} = \lambda(p)^2 \cdot \langle x, y\rangle, \, \forall p \in S_1 \text{ and } x, y \in T_p(S_1).$$

(v) Examples of equiareal and conformal maps.

(a) Given $\lambda > 0$, the linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $A = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$ is equiareal, but not conformal.

(b) Given $\lambda > 0$, the linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ is conformal, but not equiareal.

(c) The stereographic projection $\pi_N : S^2 \setminus \{N\} \to \mathbb{R}^2$ is conformal, but not equiareal.

(vi) A diffeomorphism $f : S_1 \to S_2$ is an isometry if, and only if, it is equiareal and conformal.

3 The curvature of a surface

3.1 Gaussian curvature

(i) Let S be an oriented surface with an orientation $N : S \to \mathbb{R}^3$.

(i) The *Gauss map* is the function N regarded as function $S \to S^2$.

(ii) For each $p \in S$, the *Weingarten map* is the linear map

$$W_p = -dN_p : T_p(S) \to T_P(S).$$

(iii) For each $p \in S$,

$$K(p) = \text{Det}(W_p) \text{ and } H(p) = \frac{1}{2}\text{Trace}(W_p)$$

are respectively called the *Gaussian curvature* and the *mean curvature* of S at p.

(ii) Let S be an oriented surface, and $p \in S$. Then the Weingarten map W_p is represented by a symmetric matrix with respect to any orthonormal basis of $T_p(S)$.

(iii) Examples of Gaussian and mean curvatures.

(i) Let S be a two-dimensional subspace of $\mathbb{R}^3$. Since S can be oriented by a constant unit normal field N, $W_p(v) = 0$, for each $p \in S$ and each $v \in T_p(S)$. Thus, the

$$K(p) = H(p) = 0, \forall p \in S.$$

(ii) The sphere $S^2(r)$ of radius r has an orientation given by $N(p) = p/r$, for all $p \in S^2(r)$. Consequently,

$$W_p = -\frac{1}{r}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ K(p) = \frac{1}{r^2}, \text{ and } H(p) = -\frac{1}{r}.$$

(iii) Let $u \subset \mathbb{R}^2$ be open and $f : U \to \mathbb{R}$ be a smooth function. Then G_f is an orientable surface with an orientation $F : G_f \to \mathbb{R}^3$ given by

$$F(p) = \frac{(-f_x(p), -f_y(p), 1)}{\sqrt{f_x^2(p) + f_y^2(p) + 1}}, \forall p \in G_f.$$

Let $q = (x_0, y_0)$ be critical point of f, and let $p = f(q)$. Then

$$W_p = \begin{pmatrix} f_{xx}(q) & f_{xy}(q) \\ f_{yx}(q) & f_{yy}(q) \end{pmatrix},$$

and

$$K(p) = f_{xx}(q) f_{yy}(q) - f_{xy}(q)^2.$$

Thus, if $K(p) > 0$, the p is a local extremum, and if $K(p) < 0$, then p is a saddle point.

3.2 The second fundamental form

(i) A *quadratic form* Q_T associated with a self-adjoint linear map $T : V \to V$ is given by $Q_T : V \to \mathbb{R} : v \overset{Q_T}{\longmapsto} \langle v, T(v) \rangle$.

(ii) Let S be a oriented regular surface, and let $p \in S$. Let $\{v_1, v_2\}$ be an orthonormal basis of $T_p(S)$ with respect to which $W_p = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$.

(a) The eigenvectors $\pm v_1$ and $\pm v_2$ are called the *principal directions* of S at p.

(b) The eigenvalues k_1 and k_2 are called the *principal curvatures* of S at p. If $k_1 = k_2$, then p is called an *umbilical point*.

(c) The quadratic form associated to W_p is called the *second fundamental form* II_p of S at p, that is:

$$II_p(v) = \langle W_p(v), v \rangle = \langle -dN_p(v), v \rangle.$$

(d) If $v \in T_p(S)$ with $\|v\| = 1$, then $II_p(v)$ is called the *normal curvature* of S at p in the direction of v.

(iii) Let $\{v_1, v_2\}$ be an orthonormal basis of eigenvectors of a self-adjoint linear map T corresponding to eigenvalues $\lambda_1 \leq \lambda_2$. Then

$$Q_T(\cos(\theta) v_1 + \sin(\theta) v_2) = \lambda_1 \cos^2(\theta) + \lambda_2 \sin^2(\theta).$$

In particular, λ_1 and λ_2 are the maximum and minimum values (resp.) of Q_T on S^1.

(iv) By (iii), the action of II_p on an arbitrary unit tangent vector is given by

$$II_p(\cos(\theta) v_1 + \sin(\theta) v_2) = k_1 \cos^2(\theta) + k_2 \sin^2(\theta).$$

In particular, k_1 and k_2 are the minimum and maximum normal curvatures. Moreover,

$$K(p) = k_1 k_2 \text{ and } H(p) = \frac{1}{2}(k_1 + k_2).$$

(v) Curvature of the cylinder: Consider the cylinder of radius r about the z-axis given by

$$C(r) = \{(r\cos(\theta), r\sin(\theta), z) \in \mathbb{R}^3 : \theta \in [0, 2\pi), z \in \mathbb{R}\}.$$

At $p_0 = (r\cos(\theta_0), r\sin(\theta_0), z_0) \in C(r)$, we have

$$W_{p_0} = \begin{pmatrix} -1/r & 0 \\ 0 & 0 \end{pmatrix}.$$

So, its principal directions are

$$v_1 = (-\sin(\theta_0), \cos(\theta_0), 0) \text{ and } v_2 = (0, 0, 1),$$

and its principal curvatures are

$$k_1 = -\frac{1}{r} \text{ and } k_2 = 0.$$

Hence, $C(r)$ has constant Guassian and mean curvatures given by

$$K = 0 \text{ and } H = -\frac{1}{2r}.$$

(vi) Let S be an oriented regular surface, $p \in S$, and $v \in T_p(S)$ with $\|v\| = 1$. Consider the family of all regular curves γ in S. Consider the family $F_{p,\gamma}$ of all regular curves γ in S with $\gamma(0) = 0$ and $\gamma'(0) = v$. Then:

(a) For every curve $\gamma \in F_{p,\gamma}$, we have

$$\langle \gamma''(0), N(p) \rangle = II_p(v).$$

(b) The minimum curvature at p among curves in the family (regarded as space curves) equals $|II_p(v)|$.

(vii) Let $\kappa_n = II_p(v)$, and let γ be a unit-speed curve with $a = \gamma''(0)$. Then by (vi)(a), we have

$$a = \kappa_n \cdot N(p) + \kappa_g \cdot R_{\pi/2}(v),$$

where $R_{\pi/2}$ denotes a rotation of $T_p(S)$ by $\pi/2$, and κ_n and κ_g are scalars. The scalars κ_n (resp. κ_g) are called the *normal* (resp. *geodesic*) curvatures of γ at p. Moreover, as $\kappa = \|\gamma''(0)\|$, we have

$$\kappa^2 = \kappa_n^2 + \kappa_g^2.$$

3.3 The geometry of the Gauss map

(i) Let S be an oriented regular surface with an orientation N, and let $f : U(\subset \mathbb{R}^2) \to S$ be a local parametrization at $p \in S$ that is compatible with N. Let $\alpha(t) = f(u(t), v(t))$ be a parametrized curve on S with $\alpha(0) = p$. Then:

(a) The second fundamental form in the basis $\{f_u, f_v\}$ is given by

$$II_p(\alpha') = e(u')^2 + 2fu'v' + g(v')^2,$$

where

$$\begin{aligned} e &= -\langle N_u, f_u\rangle = \langle N, f_{uu}\rangle \\ f &= -\langle N_v, f_u\rangle = \langle N, f_{uv}\rangle = \langle N, f_{vu}\rangle = -\langle N_u, f_v\rangle \\ g &= -\langle N_v, f_v\rangle = \langle N, f_{vv}\rangle \end{aligned}$$

(b) The coefficients of the Weingarten map $W_p = (a_{ij})_{2\times 2}$ are given by the equations

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = -\frac{1}{EG - f^2}\begin{pmatrix} e & f \\ f & g \end{pmatrix}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}.$$

(c) The Gaussian curvature K is given by

$$K = \mathrm{Det}(W_p) = \frac{eg - f^2}{EG - F^2},$$

and the mean curvature is given by

$$H = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}.$$

(d) The principal curvatures k_1, k_2 are the roots of the quadratic equation

$$k^2 - 2Hk + K = 0,$$

which are given by

$$k = H \pm \sqrt{H^2 - k}.$$

(ii) Some explicit computations of Gaussian curvature.

(a) The Gaussian curvature of the torus under the parametrization

$$f(u, v) = ((a + r\cos(u))\cos(v), ((a + r\cos(u))\sin(v), r\sin(u)),$$
$$0 < u < 2\pi,\ 0 < v < 2\pi,$$

can be computed to be

$$K = \frac{\cos(u)}{r(a + r\cos(u))}.$$

From this, it follows that

$$K \begin{cases} = 0, & \text{if } u = \pi/2 \text{ or } 3\pi/2, \\ < 0, & \text{if } u \in (\pi/2, 3\pi/2), \text{ and} \\ > 0, & \text{if } u \in (0, \pi/2) \sqcup (3\pi/2, 2\pi). \end{cases}$$

(b) Consider the surface of revolution as in 2.2 (viii)(a), where f and g are replaced by φ and ψ, respectively, so that

$$f(u, v) = (\varphi(v)\cos(u), \varphi(v)\sin(u), \psi(v)),$$
$$0 < u < 2\pi,\ a < v < b,\ \varphi(v) \neq 0.$$

Assuming that the rotating curve is parametrized by arc length, we have

$$G = (\varphi')^2 + (\psi')^2 = 1$$

so that

$$K = -\frac{\psi'(\psi'\varphi'' - \psi''\varphi')}{\varphi}.$$

(c) Let $G_f = \{(x, y, f(x, y)) : (x, y) \in U\}$ be the graph of a smooth function $f : U(\subset \mathbb{R}^2) \to \mathbb{R}^3$, where U is open. Then

$$K = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}$$

$$2H = \frac{(1 + f_x^2) f_{yy} - 2 f_x f_y f_{xy} + (1 + f_y^2) f_{xx}}{(1 + f_x^2 + f_y^2)^{3/1}}$$

(iii) Let S be an oriented surface, and let $p \in S$ with $K(p) \neq 0$. Then there exists a neighborhood of p in S restricted to which the Gauss map $N : S \to S^2$ is a diffeomorphism onto its image. Furthermore, $K(p)$ is positive (resp. negative) if, and only if, this diffeomorphism is orientation-preserving (resp. reversing) with respect to the given orientation N of S.

(iv) Let S be a regular surface, and let $p \in S$.

(a) If $K(p) > 0$, then a sufficiently small neighborhood of p in S lies entirely on one side of the plane $p + T_p(S)$.

(b) If $K(p) < 0$, then every neighborhood of p in S intersects both sides of the plane $p + T_p(S)$.

3.4 Minimal surfaces

(i) A regular parametrized surface is called *minimal* if its mean curvature vanishes everywhere.

(ii) Let $f : U(\subset \mathbb{R}^2) \to \mathbb{R}^3$ be a regular parametrized surface. Let $D \subset U$ be a bounded domain, and let $h : \bar{D} \to \mathbb{R}$ be a differential function. The *normal variation* of $f(\bar{D})$ determined by h, is the map

$$\varphi : \bar{D} \times (\epsilon, \epsilon) \to \mathbb{R}^3$$

defined by

$$\varphi(u, v, t) = f(u, v) + t h(u, v) N(u, v),\ (u, v) \in \bar{D} \text{ and } t \in (-\epsilon, \epsilon).$$

(iii) Given a bounded variation φ as above, for each $t \in (-\epsilon, \epsilon)$, the map $f^t : D \to \mathbb{R}^3$ given by

$$f^t(u, v) = \varphi(u, v, t)$$

is a parametrized regular surface for ϵ sufficiently small. Let E^t, F^t, G^t be the coefficients of the first fundamental form of this surface. The area of $f^t(\bar{D})$ is given by

$$A(t) = \int_{\bar{D}} \sqrt{E^t G^t - (F^t)^2}\, du\, dv = \int_{\bar{D}} \sqrt{1 - 4thH + \bar{R}}\sqrt{EG - F^2}\, du\, dv,$$

where $\bar{R} = R/(EG - F^2)$. Consequently,

$$A'(0) = -\int_{\bar{D}} 2hH\sqrt{EG - F^2}\, du\, dv.$$

(iv) Let $f : U \to \mathbb{R}^3$ be a regular parametrized surface, and let $D \subset U$ be a bounded domain. Then f is minimal if, and only if, $A'(0) = 0$ for all D and all bounded variations of $f(\bar{D})$.

(v) A regular parametrized surface $f(u, v)$ is said to be *isothermal* if

$$\langle f_u, f_v\rangle = \langle f_v, f_u\rangle \text{ and } \langle f_u, f_v\rangle = 0.$$

(vi) The *mean curvature vector* $\mathcal{H}$ of a regular parametrized surface is defined by $\mathcal{H} = HN$.

(vii) Let $f = f(u, v)$ be a regular parametrized surface, and f be isothermal. Then

$$f_{uu} + f_{vv} = 2\lambda^2\mathcal{H},$$

where $\lambda^2 = \langle f_u, f_u\rangle = \langle f_v, f_v\rangle$.

(viii) Let $f(u, v) = x(u, v), y(u, v), z(u, v))$ be a parametrized surface with isothermal coordinates. Then f is minimal if, and only if, its coordinated functions $x(u, v), y(u, v),$ and $z(u, v)$ are harmonic.

(ix) Examples of minimal surfaces.

(a) The *catenoid* given by

$$f(u, v) = (a\cosh(v)\cos(u), a\cosh(v)\sin(u), av),\ u \in (0, 2\pi), v \in \mathbb{R},$$

is the surface obtained by rotating the *catenary* $y = a\cosh(z/a)$ about the z-axis is a minimal surface. In fact, it is the only minimal surface of revolution.

(b) The *helicoid* given by

$$f(u, v) = (a\sinh(v)\cos(u), a\sinh(v)\sin(u), au),\ u \in (0, 2\pi), v \in \mathbb{R},$$

is a minimal surface.

4 The Gauss-Bonnet Theorem

4.1 Geodesics

(i) A regular curve $\gamma : I \to S$ in a surface S is called a *geodesic* if for every $t \in I$, the the acceleration vector $\gamma''(t)$ is a normal vector to S at $\gamma(t)$ (i.e normal to $T_{\gamma(t)}(S)$).

(ii) (Existence and uniqueness of geodesics) Let S be a regular surface, $p \in S$, and $v \in T_p(S)$ with $r = |v| \neq 0$. Then there exists $\epsilon = \epsilon(p, r) > 0$ such that:

(a) There exists a geodesic $\gamma_v : (-\epsilon, \epsilon) \to S$ satisfying conditions $\gamma_v(0) = p$ and $\gamma_v'(0) = v$.

(b) Any two geodesics with this domain satisying these initial conditions must be equal.

Furthermore, $\gamma_v(t)$ depends smoothly on p, v, t.

(iii) Examples of geodesics.

(a) A regular curve γ in $\mathbb{R}^2$ if, and only if, it is (or part of) a straight line parametrized by constant speed.

(b) There exists no geodesic between any two points on either side of the origin in $\mathbb{R}^2 \setminus \{(0, 0)\}$.

(c) Any geodesic on the unit sphere S^2 centered at the origin is a great circle (or a part of a great circle). That is, given $p \in S^2$ and $v \in T_p(S^2)$,

$$\gamma(t) = (\cos(t))p + (\sin(t))v$$

is a geodesic.

(d) The helix $\gamma(t) = (\cos(t), \sin(t), ct)$ is a geodesic in the cylinder $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$.

(iv) (Clairaut's theorem) Let S be a surface of revolution. Let $\beta : I \to S$ be a unit-speed curve in S . For every $s \in I$, let $\rho(s)$ denote the distance from $\beta(s)$ to the axis of rotation and let $\psi(s) \in [0, \pi]$ denote the angle between $\beta'(s)$ and the longitudinal curve through $\beta(s)$.

(a) If β is a geodesic, then $\rho(s) \sin(\psi(s))$ is constant on I.

(b) If $\rho(s)\sin(\psi(s))$ is a constant on I, then β is a geodesic, provided no segment of β equals a subsegment of a latitudinal curve.

4.2 The Local Gauss-Bonnet theorem

(i) Let S be an oriented surface.

  (a) A subset $R \subset S$ is called a *region* of S if it equals the union of a open set in S together with its boundary.

  (b) A region $R \subset S$ is called *regular* if its boundary (∂R) equals the union of the finitely many piecewise-regular simple closed curves.

  (c) A parametrization $\gamma : [a, b] \to R$ of a boundary component regular region in S is said to be *positively-oriented* if R is to one's left as one traverses γ.

(ii) (The Local Gauss-Bonnet Theorem) Let S be an oriented regular surface and $R \subset S$ a polygonal region. Let $\gamma : [a, b] \to R$ be a unit-speed positively-oriented parametrization of ∂R, with signed angles denoted by $\{\alpha_i\}$. Then

$$\underbrace{\int_a^b \kappa_g(t)\,dt + \sum_i \alpha_i}_{\text{angle displacement around }\gamma} = 2\pi - \iint_R K\,dA.$$

4.3 The Global Gauss-Bonnet Theorem

(i) Let S be a regular surface.

  (a) A *triangle* in S is a polygonal region with three vertices. The three smooth segments of the boundary of a triangular region are called *edges*.

  (b) A triangulation of a regular region $R \subset S$ means a finite family $\{T_1 \ldots, T_F\}$ of te triangles such that:

    (1) $\cup_i T_i = R$, and

    (2) if $i \neq j$, then $T_i \cap T_j = \emptyset$, or $T_i \cap T_j$ is a common edge, or $T_i \cap T_j$ is a common vertex.

(c) The *Euler characteristic* of a triangulation $\{T_1 \ldots, T_F\}$ of R is

$$\chi = V - E + F.$$

(ii) Let S be a regular surface, and $R \subset S$ be a regular region. Two distinct triangulations of R has the same Euler characteristic.

(iii) Every regular surface admits a triangulation.

(iv) The *Euler characteristic* $\chi(S)$ of a regular surface S is defined to be the Euler characteristic of any triangulation of S.

(v) The Euler characteristic of (the triangulation of) a regular surface is a topological invariant. That is, homeomorphic regular surfaces have the same Euler characteristic.

(vi) Examples of χ for surfaces.

(a) $\chi(S^2) = 2$.

(b) $\chi(D^2) = 1$, where D^2 is a closed disk. Consequently, $\chi(R) = 1$, when R is a simple polygonal region.

(c) $\chi(A) = 0$, where A is the annulus (or the cylinder).

(d) $\chi(S_g) = 2 - 2g$, where S_g denoted the closed oriented surface of genus $g \geq 1$. In particular, $\chi(S_g) < 0$, when $g \geq 2$.

(vii) (The Global Gauss-Bonnet Theorem) Let S be an oriented regular surface and $R \subset S$ a regular region with unit-speed positively-oriented boundary components. Then

$$\int_a^b \kappa_g(t)\, dt + \sum_i \alpha_i = 2\pi\chi(R) - \iint_R K\, dA,$$

where $\int_a^b \kappa_g(t)$ denotes the sum of the integrals over all boundary components of R, and $\sum \alpha_i$ denotes the sum of the signed interior angles over all vertices of all boundary components of R.

4.4 Some applications of the Gauss-Bonnet theorem

(i) If S is a closed oriented surface, then

$$\iint_S K\,dA = 2\pi\chi(S).$$

In particular, if S has constant curvature K, then

$$K\,Area(S) = 2\pi\chi(S).$$

(ii) If S is a regular surface with $K \geq 0$, then two geodesics from a point $p \in S$ cannot again at a point $q \in S$ so that they cobound a region that is diffeomorphic to a disk.

(iii) If S is a regular surface that is diffeomorphic to a cylinder with $K < 0$, then S has at most one closed geodesic (up to reparametrization).

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